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# About the Conditions for the Rees Ring to Be C-M or Gorenstein (代数幾何学への可換環論の応用)

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About the conditions for the Rees ring  
to be C-M or Gorenstein.

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Let  $A$  be a Noetherian local ring of dimension  $d$  and  $\mathfrak{m}$   
the maximal ideal of  $A$ . For an ideal  $\mathfrak{q}$  of  $A$  we put

$$R(\mathfrak{q}) = \bigoplus_{n \geq 0} \mathfrak{q}^n$$

and call it the Rees algebra of  $A$  relative to  $\mathfrak{q}$ . We identified  
with the subring  $A[\{aT ; a \in \mathfrak{q}\}]$  of the polynomial ring  $A[T]$ .

Now we consider the following question:

Question. Is a given property of the ring  $A$  inherited by the  
Rees algebra of  $A$ ?

For example, Noetherian, integral domain, integrally closed,  
regular local ring, C-M ring, Gorenstein ring and etc.

Here we will examine the property of being C-M or Gorenstein.  
Concerning this property, the following results are well known.

J. Barshay ([1]) (1973).

① If  $A$  is C-M,  $a_1, \dots, a_s$  an  $A$ -sequence and  $\mathfrak{q} = (a_1, \dots, a_s)$ .  
Then  $R(\mathfrak{q})$  is C-M.

② If  $A$  is C-M,  $a_1, a_2$  an  $A$ -sequence and  $\mathfrak{q} = (a_1, a_2)^n$ ,  
where  $n$  is a positive integer. Then  $R(\mathfrak{q})$  is C-M.

Moreover, J. Barshay presented in his paper the following.

Let  $A$  be a C-M ring,  $a_1, \dots, a_d$  a system of parameters,  
 $n$  a positive integer and  $\mathfrak{q} = (a_1, \dots, a_d)^n$ . Then is  $R(\mathfrak{q})$  a  
C-M ring?

G. Valla ([9]) (1976) has solved the above problem and

S. Goto ([2]) (1978) has given a shorter proof for this one.

Remark. We note that the converse of Barshay's problem is not true in general. In fact, Hochster-Roberts ([5]) have given an example:

Let  $A = k[x^2, x^3, y, xy]$  and  $\underline{q} = (x^2, y)$ . Then  $R(\underline{q})$  is a Gorenstein ring, but  $A$  is not a C-M ring. ( $A$  is a Buchsbaum ring).

Thus the following problem seems to be interesting.

Problem. Find the conditions for  $R(\underline{q})$  to be C-M or Gorenstein.

At first, as a partial answer

Proposition. If  $A$  is a local integral domain of dim 2,  $a, b$  a system of parameters for  $A$  and  $\underline{q} = (a, b)$ . Then

(i)  $R(\underline{q})$  is C-M if and only if  $(a) : b \cap (b) : a \subseteq (a) \cap (b)$ .

In particular,  $R(\underline{q})$  is C-M for every parameter ideal  $\underline{q}$  if and only if  $A$  is Buchsbaum.

(ii) The following statements are equivalent.

1.  $A$  is Gorenstein.
2.  $R(\underline{q})$  is Gorenstein for some parameter ideal  $\underline{q}$ .
3.  $R(\underline{q})$  is Gorenstein for every parameter ideal  $\underline{q}$ .

For a general case, there is the following result.

S. Goto and Y. Shimoda ([3]).

Proposition. For a Noetherian local ring  $A$ , the following two conditions are equivalent.

1. The Rees algebra  $R(\underline{q})$  is C-M for every parameter ideal  $\underline{q}$
2.  $A$  is a Buchsbaum ring and the local cohomology module

$$H_{\underline{m}}^i(A) = (0) \text{ for } i \neq 1, d.$$

Thus we have

Corollary. Let  $A$  be a Noetherian local ring and assume that  $\text{depth } A \neq 1$ . Then  $A$  is C-M if and only if so is  $R(\underline{q})$  for every parameter ideal  $\underline{q}$ .

By the above result, we have done to find the conditions

that  $R(\underline{q})$  is a C-M ring for every parameter ideals  $\underline{q}$ . Now let us find the conditions that  $R(\underline{q})$  is a C-M ring for a parameter ideal  $\underline{q}$ . A complete answer is

**Theorem 1 (Goto-Shimoda).** Let  $0 = s = d$  be an integer,  $a_{s+1}, \dots, a_d$  a subsystem of parameters for  $A$  and  $\underline{q} = (a_{s+1}, \dots, a_d)$ . Then the following conditions are equivalent.

- (1)  $R(\underline{q})$  is a C-M ring
- (2) the following four properties are satisfied.
  - (i)  $A/\underline{q}^n$  is a C-M ring for every integer  $n > 0$ .
  - (ii)  $H_{\underline{m}}^i(A) = (0)$  for  $i \neq s+1, d$
  - (iii)  $\underline{q}H_{\underline{m}}^{s+1}(A) = (0)$  if  $s+1 < d$
  - (iv)  $\text{Hom}_A(H_{\underline{m}}^{s+1}(A), E(A/\underline{m}))$  is an C-M  $A$ -module of dimension  $s$  if  $s+1 < d$  and  $H_{\underline{m}}^{s+1}(A) \neq (0)$ .

By the above theorem, we immediately have the followings:

**Corollary 1. ([8])** Under the same situation of Theorem 1,  $A$  is a C-M ring if and only if so is  $R(\underline{q})$  for every ideal  $\underline{q}$  generated by a subsystem of parameters of length  $s$ .

**Corollary 2.** We put  $r = d - s$  and denote by  $r(A)$  the type of a C-M ring  $A$ . Then we have

$$r(R(\underline{q})) = \begin{cases} (r-1) \cdot \mu_A(K_A) + \mu_A(\text{Hom}(H_{\underline{m}}^{s+1}(A), E(A/\underline{m}))) & (r \geq 2) \\ r(A) & (r = 1) \end{cases}$$

Therefore,  $R(\underline{q})$  is a Gorenstein ring if and only if so is  $A$  and  $r \leq 2$ .

Now we will consider the case of the Rees algebra relative to maximal ideal  $\underline{m}$ , that is,  $R(\underline{m})$ . At first we denote

$$G(\underline{m}) = \bigoplus_{n=0}^{\infty} \underline{m}^n / \underline{m}^{n+1}$$

and call it the associated graded ring. The canonical map  $f : \text{Proj } R(\underline{m}) \longrightarrow \text{Spec}(A)$  is the blowing up of  $\text{Spec}(A)$  with center  $\text{Spec}(A/\underline{m})$  and the fibre of  $f$  at the closed point is isomorphic to  $\text{Proj}(G(\underline{m}))$ . Hence the properties of  $\text{Proj}(G(\underline{m}))$  determine some of properties of  $\text{Proj}(R(\underline{m}))$ . For example, concerning the property of being C-M or Gorenstein we have

Hochster-Ratliff, Jr. (1973)

Suppose that  $A$  is C-M (resp. Gorenstein). If  $G(\underline{m})$  is C-M (resp. Gorenstein), then so is  $\text{Proj}(R(\underline{m}))$ . Thus, to show that  $\text{Proj}(R(\underline{m}))$  is C-M (resp. Gorenstein) it suffices to show that  $G(\underline{m})$  itself is C-M (resp. Gorenstein). Now about the question whether  $G(\underline{m})$  is C-M or Gorenstein the following results are well known.

J. Sally ([6]) (1978)

(i) If  $A$  is a C-M ring of dimension  $d$  and embedding dimension  $v(A) = e(A) + d - 1$ , then  $G(\underline{m})$  is a C-M ring.

(ii) If  $A$  is a Gorenstein ring of dimension  $d$  and  $v(A) = e(A) + d - 2$ , then  $G(\underline{m})$  is a Gorenstein ring.

Thus we have partially done to solve the question whether  $\text{Proj}(R(\underline{m}))$  is C-M or Gorenstein in the above sense. But the question whether  $R(\underline{m})$  itself is C-M or Gorenstein still remains to be unsolved as far as I know. Here let us discuss about this problem. A complete answer is

Theorem 2. Suppose that  $A$  is a C-M ring. Then the following conditions are equivalent.

- (1)  $R(\underline{m})$  is a C-M ring
- (2)  $G(\underline{m})$  is a C-M ring and  $a(G(\underline{m})) < 0$ .

Theorem 3. Suppose that  $A$  is a C-M ring with  $\dim A \geq 2$ . Then the following two conditions are equivalent.

- (1)  $R(\underline{m})$  is a Gorenstein ring  
 (2)  $G(\underline{m})$  is a Gorenstein ring and  $a(G(\underline{m})) = -2$ .

Here  $a(G(\underline{m}))$  denotes the invariant of  $G(\underline{m})$  as follows:

Let  $R = \bigoplus_{n \geq 0} R_n$  be a Noetherian graded ring with  $R_0$  a field

and  $M$  the irrelevant maximal ideal of  $R$ . We put  $\dim R = r$ .

we define

$$a(R) = \max \left\{ k \in \mathbb{Z} : [H_{\underline{M}}^r(R)]_k \neq (0) \right\}.$$

Notice that if  $R$  is a C-M ring and if  $h_1, \dots, h_s$  is a homogeneous  $R$ -regular sequence, then

$$a(R) = a(R/(h_1, \dots, h_s)) - \sum_{i=1}^s \deg h_i.$$

In the above theorems, as we can replace  $A[U]_{\underline{m}A[U]}$

instead of  $A$ , we may assume that  $A/\underline{m}$  is an infinite residue field. Then it is well known that there exist  $a_1, \dots, a_d$  in  $\underline{m}$  such that  $\underline{m}^{t+1} = (a_1, \dots, a_d)\underline{m}^t$  for some  $t \geq 0$ . Therefore we may restate the condition (2) of Theorem 2 (resp. Theorem 3) as follows:

- (2)'  $G(\underline{m})$  is a C-M (resp. Gorenstein) ring and  $\underline{m}^d = (a_1, \dots, a_d)\underline{m}^{d-1}$  (resp.  $\underline{m}^{d-1} = (a_1, \dots, a_d)\underline{m}^{d-2}$  and  $(a_1, \dots, a_d) \not\subset \underline{m}^{d-2}$ ).

In case  $\dim A = 2$ , we have

Proposition. Suppose that  $\dim A = 2$ . Then

$R(\underline{m})$  is C-M (resp. Gorenstein) if and only if  $A$  is C-M and  $v(A) = e(A) + 1$  (resp.  $A$  is regular).

Now we will give some examples satisfying the above theorems.

1. If  $A$  is a rational surface singularity of  $\dim A = 2$ , then  $A$  satisfies (2) of the above proposition and hence  $R(\underline{m})$  is C-M.

2. Suppose that  $A$  is C-M and  $\dim A = 3$ . Then  $R(\underline{m})$  is a Gorenstein ring if and only if  $A$  is abstract hypersurface (i.e.,  $A$  is hypersurface) and  $e(A) = 2$ .

3. Let  $d > 0$  be an integer. We put

$$A = k[[X_1, \dots, X_d]^d].$$

Then we have  $G(\underline{m}) = k[(X_1, \dots, X_d)^d]$  and  $\underline{m}^d = (X_1^d, \dots, X_d^d)\underline{m}^{d-1}$ .

Thus  $G(\underline{m})$  is a Gorenstein ring and  $a(G(\underline{m})) = -1$ . Hence  $R(\underline{m})$  is a C-M ring.

4. Let  $n, r$  be integers with  $r + 2 = n$  and  $P = k[[X_1, \dots, X_n]]$ .

Let  $f_1, \dots, f_r$  be a homogeneous  $P$ -regular sequence. We put  $A = P/(f_1, \dots, f_r)$ . Then  $G(\underline{m}) = k[X_1, \dots, X_n]/(f_1, \dots, f_r)$  is a Gorenstein ring and

$$\begin{aligned} a(G(\underline{m})) &= a(k[X_1, \dots, X_n]) + \sum_{i=1}^r \deg f_i \\ &= -n + \sum_{i=1}^r \deg f_i \end{aligned}$$

Thus  $R(\underline{m})$  is C-M (resp. Gorenstein) if and only if

$$\sum_{i=1}^r \deg f_i < n \text{ (resp. } \sum_{i=1}^r \deg f_i = n - 2).$$

5. Let  $e \geq 3$  and  $d > 0$  be integers. Put  $P = k[[X, Y_1, \dots, Y_{d-1}]]$

and  $A = k[[X^e, X^{e+1}, \dots, X^{2e-2}, Y_1, \dots, Y_{d-1}]]$ . Then  $A$  is Gorenstein

with  $\dim A = d$ ,  $e(A) = e$  and  $v(A) = e + d - 2$ . By virtue of

the result in J. Sally's paper, we can see that  $G(\underline{m})$  is Gorenstein and  $\underline{m}^3 = \underline{q}\underline{m}^2$  ( $\underline{q} \nmid \underline{m}^2$ ). Hence if  $d \geq 3$ ,  $R(\underline{m})$  is C-M and

$R(\underline{m})$  is Gorenstein if and only if  $d = 4$ .

Remark. Concerning the properties of being complete intersection or regular we have

- (i)  $R(\underline{m})$  is complete intersection if and only if  $A$  is regular and  $\dim A \leq 2$ .
- (ii)  $R(\underline{m})$  is regular if and only if  $A$  is D.V.R.

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